Filomat 30:1 (2016), 29–39
DOI 10.2298/FIL1601029A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by Modified Integral Type Jakimovski-Leviatan Operators

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Abstract. In this paper, we give a generalization of the integral type Jakimovski-Leviatan operators, introduced by Ciupa [1]. Theorems on convergence and the rate of convergence of the operators by using the first and second order modulus of continuity are established. We also study the convergence of these operators in a weighted space of functions.

1. Introduction

The Favard-Szász operators are important and have been studied intensively, because they play a very important role in different branches of analysis, such as numerical analysis, approximation theory and so on. Let f be real-valued functions on $[0,\infty)$ and satisfy the property $\left|f(x)\right| \leq \beta e^{\alpha x}$ for some finite constants $\alpha, \beta > 0$ and denote the set of functions that satisfy this inequality by $E\,[0,\infty)$. The Favard-Szász operators are defined as follows:

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \quad n \geq 1,\ x \in [0,\infty). \tag{1.1}$$

The operator $S_n(f;x)$ convergence to $f(x)$ at each point $x \geq 0$ as $n \to \infty$. This result was proved by Szász in [10].

Later in 1969, Jakimovski and Leviatan [8] introduced a Favard–Szász type operator by means of Appell polynomials. Let $g(u) = \sum_{n=0}^{\infty} a_n u^n$ be an analytic function in the disc $|u| < r$, $(r > 1)$ with $g(1) \neq 0$ and the Appell polynomials $p_k(x) = \sum_{i=0}^{k} a_i \frac{x^{k-i}}{(k-i)!}$, $(k \in \mathbb{N})$ are defined by the generating functions

$$g(u)e^{ux} \equiv \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1.2}$$

2010 *Mathematics Subject Classification*. Primary 41A17, 41A35, 41A36
Keywords. Jakimovski-Leviatan operators, Favard-Szász operators, Appell polynomials, modulus of continuity, weighted space.
Received: 11 February 2014; Accepted: 23 August 2014
Communicated by Dragan S. Djordjević
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The Jakimovski–Leviatan operators P_n associate to each function $f \in E[0,\infty)$

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad x \geq 0. \tag{1.3}$$

For all sufficiently large n, the operators P_n are well–defined, since the infinite sum in (1.3) is convergent if $n > \alpha/logr$. Wood [12] proved that the operator P_n is positive on $[0,\infty)$ if and only if $\frac{a_n}{g(1)} \geq 0$ for $n \in \mathbb{N}$. In [8], Jakimovski and Leviatan established the analogue result of Szász and proved that, for all $f \in E[0,\infty)$,

$$\lim_{n\to\infty} P_n(f;x) = f(x)$$

the convergence being uniform in each compact subset of $[0,\infty)$.

Note that, in the special case $g(u) = 1$ in (1.3), we get $p_k(x) = \dfrac{x^k}{k!}$, and we recover the well-known classical Favard-Szász operators defined by (1.1).

The generalized Favard-Szász type operators (see [6-11]) were defined with

$$S_n(f;a_n,b_n;x) = e^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{k!} f\left(\frac{k}{b_n}\right), \; x \in [0,\infty), \; n \in \mathbb{N} \tag{1.4}$$

where $\{a_n\}$ and $\{b_n\}$ are given increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{1}{b_n} = 0, \; \frac{a_n}{b_n} = 1 + O(\frac{1}{b_n}).$$

The case $a_n = b_n = n$ yields the classical operators of (1.1). In [11], the author studied approximation properties of the operators (1.4) in polynomial weighed spaces. In [6], the authors defined the weighted modulus of continuity and obtained the rate of convergence of the operators (1.4) to f function on all positive semi-axis.

Let $f \in L_1[0,\infty)$. In [1], Ciupa introduced and studied the integral type operators, generalization of the operators (1.3) as follows:

$$L_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t) dt. \tag{1.5}$$

In case $g(u) = 1$ and $\lambda = 0$, one can have the operators defined by Mazhar and Totik [9],

$$S_n^*(f;x) = n \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^k}{k!} f(t)\, dt.$$

In this paper, inspired by the operators (1.4), we define the generalization of (1.5) as follows

$$L_n^*(f;x) = \frac{e^{-a_n x}}{g(1)} \sum_{k=0}^{\infty} p_k(a_n x) \frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-b_n t} t^{\lambda+k} f(t) dt \tag{1.6}$$

where $\{a_n\}$ and $\{b_n\}$ are given increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{1}{b_n} = 0, \; \frac{a_n}{b_n} = 1 + O(\frac{1}{b_n}). \tag{1.7}$$

The purpose of this paper is to study the some approximation properties of the operators (1.6). Now, we give some definitions in order to establish the next results.

By $C_B[0,\infty)$,we denote the class on real valued continuous bounded functions $f(x)$ for $x\in[0,\infty)$ with the norm

$$\left\|f\right\|_{C_B[0,\infty)}=\sup_{x\in[0,\infty)}\left|f(x)\right|.$$

For $\delta>0$, the first and second order modulus of continuity of $f\in C_B[0,\infty)$ are defined as

$$\omega(f,\delta)=\sup_{\substack{0<h\leq\delta\\ x\in[0,\infty)}}\left|f(x+h)-f(x)\right| \tag{1.8}$$

and

$$\omega_2(f,\delta)=\sup_{\substack{0<h\leq\delta\\ x\in[0,\infty)}}\left|f(x+2h)-2f(x+h)+f(x)\right| \tag{1.9}$$

respectively. We also denote by $C_B^2[0,\infty)=\{f\in C_B[0,\infty): f',f''\in C_B[0,\infty)\}$ with the norm

$$\left\|f\right\|_{C_B^2[0,\infty)}=\left\|f\right\|_{C_B[0,\infty)}+\left\|f'\right\|_{C_B[0,\infty)}+\left\|f''\right\|_{C_B[0,\infty)}.$$

We consider the weighted spaces of the functions which are defined on the semi-axis $[0,\infty)$ and satisfy the inequality $\left|f(x)\right|\leq M_f\rho(x)$. Here $\rho(x)=x^2+1$ is a weight function and M_f is a constant depending only on f. We denote the set of functions that satisfy this inequality by $B_\rho[0,\infty)$. By $C_\rho[0,\infty)$, we denote the subspace of all continuous functions belonging to $B_\rho[0,\infty)$. Also let $C_\rho^*[0,\infty)$ be the subspace of all functions $f\in C_\rho[0,\infty)$, for which $\lim_{x\to\infty}\frac{f(x)}{\rho(x)}=k_f<\infty$. Obviously $C_\rho[0,\infty)$ is a linear normed space with the $\rho-$norm:

$$\left\|f\right\|_\rho=\sup_{x\in[0,\infty)}\frac{\left|f(x)\right|}{\rho(x)}.$$

It is well-known that the modulus of continuity and smoothness given by (1.8) and (1.9) in general do not tend to zero with $\delta\to0$ on $[0,\infty)$, so we use following the weighted modulus of continuity [7]:

$$\Omega(f;\delta)=\sup_{\substack{|t-x|\leq\delta\\ x,t\in[0,\infty)}}\frac{\left|f(t)-f(x)\right|}{\left[1+(t-x)^2\right]\rho(x)}. \tag{1.10}$$

The following Lemma give some properties of Ω.

Lemma 1.1 ([7]). *Let $f\in C_\rho^*[0,\infty)$ .Then we have*

$$i)\quad \lim_{\delta\to0}\Omega(f;\delta)=0, \tag{1.11}$$

for each $\lambda>0$

$$ii)\ \Omega(f;\lambda\delta)\leq2(1+\lambda)(1+\delta^2)\Omega(f;\delta).$$

From this inequality, for $x,t\in[0,\infty)$ we get

$$\left|f(t)-f(x)\right|\leq2\left(1+\frac{1}{\delta}\left|t-x\right|\right)(1+\delta^2)(1+x^2)(1+(t-x)^2)\Omega(f;\delta). \tag{1.12}$$

2. Main Results

In this section, we will need the following lemma for proving our main results about convergence for $L_n^* f$ operators.

Lemma 2.1. *For all $x \geq 0$, we have*

$$L_n^*(1;x) = 1, \tag{2.1}$$

$$L_n^*(t;x) = \frac{a_n}{b_n}x + \frac{1}{b_n}\left(\lambda + 1 + \frac{g'(1)}{g(1)}\right), \tag{2.2}$$

$$\begin{aligned} L_n^*(t^2;x) &= \frac{a_n^2}{b_n^2}x^2 + \frac{2a_n}{b_n^2}\left(\lambda + 2 + \frac{g'(1)}{g(1)}\right)x \\ &\quad + \frac{1}{b_n^2}\left((\lambda+1)(\lambda+2) + (2\lambda+4)\frac{g'(1)}{g(1)} + \frac{g''(1)}{g(1)}\right). \end{aligned} \tag{2.3}$$

Proof. By using $g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k$, we have

$$\sum_{k=0}^{\infty} p_k(a_n x) = g(1)e^{a_n x}, \tag{2.4}$$

$$\sum_{k=0}^{\infty} k p_k(a_n x) = (g'(1) + a_n g(1)x)e^{a_n x}, \tag{2.5}$$

$$\sum_{k=0}^{\infty} k^2 p_k(a_n x) = (g''(1) + 2a_n g'(1)x + g'(1) + a_n g(1)x + a_n^2 g(1)x^2)e^{a_n x}. \tag{2.6}$$

From (2.4) and definition of L_n^* , one can easily obtain $L_n^*(1;x) = 1$. Now let us calculate the $L_n^*(t;x)$:

$$\begin{aligned} L_n^*(t;x) &= \frac{e^{-a_n x}}{g(1)} \sum_{k=0}^{\infty} p_k(a_n x) \frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-b_n t} t^{\lambda+k+1} dt \\ &= \frac{e^{-a_n x}}{g(1)} \sum_{k=0}^{\infty} p_k(a_n x) \frac{1}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-b_n t} (b_n t)^{\lambda+k+1} dt \\ &= \frac{e^{-a_n x}}{g(1)} \sum_{k=0}^{\infty} p_k(a_n x) \frac{\lambda+k+1}{b_n} \\ &= \frac{\lambda+1}{b_n} \frac{e^{-a_n x}}{g(1)} \sum_{k=0}^{\infty} p_k(a_n x) + \frac{1}{b_n} \frac{e^{-a_n x}}{g(1)} \sum_{k=0}^{\infty} k p_k(a_n x). \end{aligned}$$

By using (2.4) and (2.5), we get the desired result. Similarly we can prove equality (2.3). □

Theorem 2.2. *Let $\{L_n^*\}$ be the sequence of linear positive operators defined by (1.6). Then for each $f \in C[0,\infty) \cap E[0,\infty)$, we have*

$$\lim_{n\to\infty} L_n^*(f;x) = f(x), \quad \textit{uniformly in } x \in [0,A].$$

Proof. From Lemma 2.1, we have $\lim_{n\to\infty} \left\| L_n^*(t^i;x) - x^i \right\|_{C[0,A]} = 0$, $i = 0,1,2$ and the result follows from the well-known Bohman–Korovkin theorem. □

For $g(u) = e^u$ and $g(u) = u$, comparison of the convergence of $L_n^*(f;x)$(red) defined by (1.6) and $L_n(f;x)$(blue) defined by (1.5) to $f(x) = \sqrt{x}e^{-2x}$(black) is illustrated in Fig.1. and Fig.2., respectively where $n = 10$ and $a_n = n + \sqrt{n}$, $b_n = n + 3$.

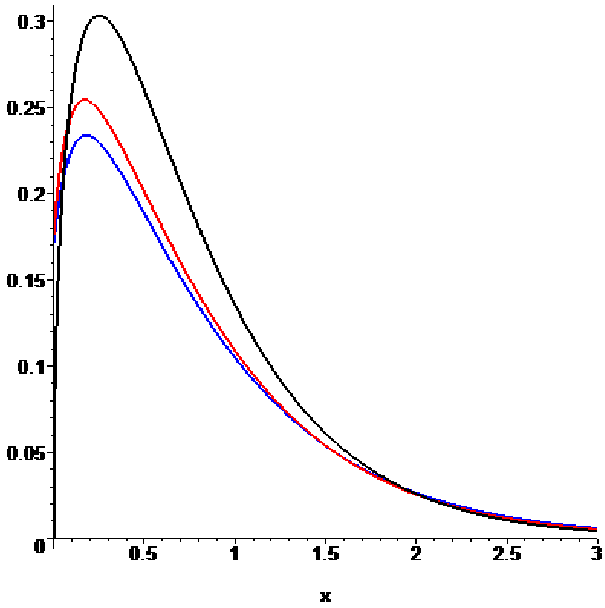

Figure 1: The convergence of $L_n^*(f;x)$ and $L_n(f;x)$ to $f(x)$ ($g(u) = e^u$).

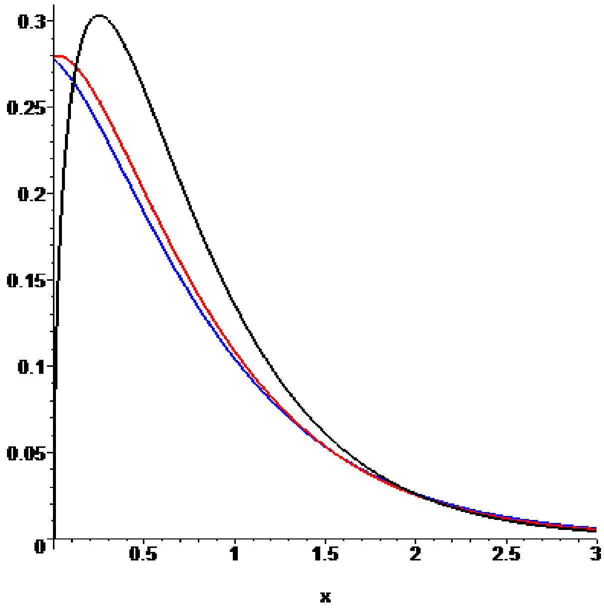

Figure 2: The convergence of $L_n^*(f;x)$ and $L_n(f;x)$ to $f(x)$ ($g(u) = u$).

Using the first and the second modulus of continuity we have the following approximation results.

Theorem 2.3. *If $f \in C[0,\infty)$, then the inequality*

$$\left|L_n^*(f;x) - f(x)\right| \leq \left\{1 + \left[2x + \frac{1}{b_n}\left((\lambda+1)(\lambda+2) + (2\lambda+4)\frac{g'(1)}{g(1)} + \frac{g''(1)}{g(1)}\right)\right]^{1/2}\right\}\omega(f;\frac{1}{\sqrt{b_n}})$$

is satisfied for a sufficiently large n.

Proof. From the definition of L_n^* and (1.8), we can write

$$\begin{aligned}\left|L_n^*(f;x) - f(x)\right| &\leq \frac{e^{-a_n x}}{g(1)}\sum_{k=0}^{\infty} p_k(a_n x)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-b_n t}t^{\lambda+k}\left|f(t) - f(x)\right| dt \\ &\leq \frac{e^{-a_n x}}{g(1)}\sum_{k=0}^{\infty} p_k(a_n x)\left(1 + \frac{1}{\delta}\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-b_n t}t^{\lambda+k}|t-x|\, dt\right)\omega(f;\delta).\end{aligned}$$

By using Cauchy- Schwarz inequality,

$$\begin{aligned}\int_0^{\infty} e^{-b_n t}t^{\lambda+k}|t-x|\, dt &\leq \sqrt{\int_0^{\infty} e^{-b_n t}t^{\lambda+k}dt}\sqrt{\int_0^{\infty} e^{-b_n t}t^{\lambda+k}(t-x)^2 dt}. \\ &\leq \frac{\Gamma(\lambda+k+1)}{b_n^{\lambda+k+1}}\sqrt{\frac{1}{b_n^2}(\lambda+k+1)(\lambda+k+2) - \frac{2}{b_n}(\lambda+k+1)x + x^2}\end{aligned}$$

So we have

$$\begin{aligned}\left|L_n^*(f;x)-f(x)\right| &\leq \omega(f;\delta)\Bigg(1+\frac{1}{\delta}\frac{e^{-a_nx}}{g(1)}\\ &\times\sum_{k=0}^{\infty}p_k(a_nx)\sqrt{\frac{(\lambda+1)(\lambda+2)}{b_n^2}+\frac{(2\lambda+3)k}{b_n^2}+\frac{k^2}{b_n^2}-\frac{2}{b_n}(\lambda+1)x-\frac{2k}{b_n}x+x^2}\Bigg).\end{aligned}$$

Now, if we use again Cauchy- Schwarz inequality we get

$$\begin{aligned}\left|L_n^*(f;x)-f(x)\right| &\leq \omega(f;\delta)\Bigg\{1+\frac{1}{\delta}\Bigg[\left(1-\frac{a_n}{b_n}\right)^2x^2+\left(\frac{2a_n}{b_n^2}\left(\lambda+2+\frac{g'(1)}{g(1)}\right)-\frac{2}{b_n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right)\right)x\\ &+\frac{1}{b_n^2}\left((\lambda+1)(\lambda+2)+(2\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)\Bigg]^{1/2}\Bigg\}.\end{aligned}$$

From condition (1.7) and choosing $\delta=\frac{1}{\sqrt{b_n}}$, one obtains desired result for sufficiently large n. □

Theorem 2.4. *If $f\in C[0,A]$, then for any $x\in[0,A]$ we have*

$$\left|L_n^*(f;x)-f(x)\right|\leq\frac{2h}{A}\left\|f\right\|+\frac{3}{4}(3+\frac{A}{h})\omega_2(f;h)$$

where

$$\begin{aligned}h &= \Bigg[\left(1-\frac{a_n}{b_n}\right)^2x^2+\left(\frac{2a_n}{b_n^2}\left(\lambda+2+\frac{g'(1)}{g(1)}\right)-\frac{2}{b_n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right)\right)x\\ &+\frac{1}{b_n^2}\left((\lambda+1)(\lambda+2)+(2\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)\Bigg]^{1/2}.\end{aligned}$$

Proof. Let f_h be the Steklov function defined by $f_h(x)=4h^{-2}\int_0^{h/2}(2f(x+u+v)-f(x+2u+2v))dudv$. From [5], we have

$$\left\|f-f_h\right\|\leq\frac{3}{4}\omega_2(f;h) \quad\text{and}\quad \left\|f_h''\right\|\leq\frac{3}{2}h^{-2}\omega_2(f;h).$$

Since $L_n^*(1;x)=1$, we can write

$$\left|L_n^*(f;x)-f(x)\right|\leq 2\left\|f-f_h\right\|+\left|L_n^*(f_h;x)-f_h(x)\right|. \tag{2.7}$$

From [5] we have

$$\left|L_n^*(f_h;x)-f_h(x)\right|\leq\left\|f_h'\right\|\sqrt{L_n^*((t-x)^2;x)}+\frac{1}{2}\left\|f_h''\right\|L_n^*((t-x)^2;x).$$

Using the results from [5] and [13], we get

$$\left\|f_h'\right\|\leq\frac{2}{A}\left\|f_h\right\|+\frac{A}{2}\left\|f_h''\right\|\leq\frac{2}{A}\left\|f\right\|+\frac{3A}{4}h^{-2}\omega_2(f;h).$$

By using this inequality and choosing $h=\sqrt{L_n^*((t-x)^2;x)}$, one obtain

$$\left|L_n^*(f_h;x)-f_h(x)\right|\leq\frac{2}{A}\left\|f\right\|h+\frac{3A}{4}h^{-1}\omega_2(f;h)+\frac{3}{4}\omega_2(f;h) \tag{2.8}$$

and if we use (2.8) in (2.7), one get

$$\left|L_n^*(f;x)-f(x)\right| \le \frac{2h}{A}\left\|f\right\| + \frac{3}{4}(3+\frac{A}{h})\omega_2(f;h).$$

Now we calculate $h=\sqrt{L_n^*((t-x)^2;x)}$. From (2.1), (2.2) and (2.3), we have

$$\begin{aligned} L_n^*((t-x)^2;x) &= \left(1-\frac{a_n}{b_n}\right)^2 x^2 + \left(\frac{2a_n}{b_n^2}\left(\lambda+2+\frac{g'(1)}{g(1)}\right) - \frac{2}{b_n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right)\right)x \\ &\quad + \frac{1}{b_n^2}\left((\lambda+1)(\lambda+2)+(2\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right) \end{aligned}$$

and we obtain desired result. □

Theorem 2.5. *For every function $f \in C_B^2[0,\infty)$, we have*

$$\left|L_n^*(f;x)-f(x)\right| \le \frac{1}{b_n}\left(x+\frac{1}{2}\left((\lambda+1)(\lambda+2)+(2\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)\right)\left\|f\right\|_{C_B^2}.$$

Proof. By using the Taylor expansion to the function $f \in C_B^2[0,\infty)$

$$L_n^*(f;x)-f(x) = f'(x)L_n^*(t-x;x) + \frac{1}{2}f''(\varphi)L_n^*((t-x)^2;x)\ \ ,\ \varphi\in(t,x)\,. \tag{2.9}$$

From Lemma 2.1, we have

$$\begin{aligned} L_n^*(t-x;x) &= \left(\frac{a_n}{b_n}-1\right)x + \frac{1}{b_n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right), \\ L_n^*((t-x)^2;x) &= \left(1-\frac{a_n}{b_n}\right)^2 x^2 + \left(\frac{2a_n}{b_n^2}\left(\lambda+2+\frac{g'(1)}{g(1)}\right) - \frac{2}{b_n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right)\right)x \\ &\quad + \frac{1}{b_n^2}\left((\lambda+1)(\lambda+2)+(2\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right). \end{aligned}$$

By using these equalities and (2.9), we get

$$\begin{aligned} \left|L_n^*(f;x)-f(x)\right| &\le \left(\left(\frac{a_n}{b_n}-1\right)x+\frac{1}{b_n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right)\right)\left\|f'\right\|_{C_B} \\ &\quad + \frac{1}{2}\left[\left(1-\frac{a_n}{b_n}\right)^2 x^2 + \left(\frac{2a_n}{b_n^2}\left(\lambda+2+\frac{g'(1)}{g(1)}\right)-\frac{2}{b_n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right)\right)x \right. \\ &\quad \left. + \frac{1}{b_n^2}\left((\lambda+1)(\lambda+2)+(2\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)\right]\left\|f''\right\|_{C_B}. \end{aligned}$$

For sufficiently large n, we obtain

$$\begin{aligned} \left|L_n^*(f;x)-f(x)\right| &\le \frac{1}{b_n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right)\left\|f'\right\|_{C_B} \\ &\quad + \frac{1}{b_n}\left(x+\frac{1}{2}\left((\lambda+1)(\lambda+2)+(2\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)\right)\left\|f''\right\|_{C_B} \\ &\le \frac{1}{b_n}\left(x+\frac{1}{2}\left((\lambda+1)(\lambda+2)+(2\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)\right)\left(\left\|f'\right\|_{C_B}+\left\|f''\right\|_{C_B}\right). \end{aligned}$$

So the proof is completed. □

Now we give a rate of convergence theorem for the operator $L_n^* f$ in the weighted space. Note that following Korovkin's type theorem and some important results in this space were proved by Gadzhiev in [2-4].

Theorem 2.6. *Let $\{T_n\}$ be the sequence of linear positive operators which are mappings from C_ρ into B_ρ satisfying the conditions*

$$\lim_{n\to\infty} \|T_n(t^\upsilon;x) - x^\upsilon\|_\rho = 0, \quad \upsilon = 0,1,2.$$

Then, for any function $f \in C_\rho^*$,

$$\lim_{n\to\infty} \left\|T_n f - f\right\|_\rho = 0,$$

and there exists a function $f^* \in C_\rho \backslash C_\rho^*$ such that

$$\lim_{n\to\infty} \left\|T_n f^* - f^*\right\|_\rho \geq 1.$$

Theorem 2.7. *Let $\{L_n^*\}$ be the sequence of linear positive operators defined by (1.6). Then for each function $f \in C_\rho^*$,*

$$\lim_{n\to\infty} \left\|L_n^*(f;x) - f(x)\right\|_\rho = 0.$$

Proof. From (2.1), clearly $\lim_{n\to\infty} \left\|L_n^*(1;x) - 1\right\|_\rho = 0$ and from (2.2), we get

$$\sup_{x\in\mathbb{R}_0} \frac{\left|L_n^*(t;x) - x\right|}{1+x^2} \leq \left|\frac{a_n}{b_n} - 1\right| \sup_{x\in\mathbb{R}_0} \frac{x}{1+x^2} + \frac{1}{b_n}\left|\lambda + 1 + \frac{g'(1)}{g(1)}\right| \sup_{x\in\mathbb{R}_0} \frac{1}{1+x^2}.$$

Hence we obtain

$$\lim_{n\to\infty} \left\|L_n^*(t;x) - x\right\|_\rho = 0.$$

Also, by using (2.3) we can write

$$\begin{aligned} \sup_{x\in\mathbb{R}_0} \frac{\left|L_n^*(t^2;x) - x^2\right|}{1+x^2} \quad \leq \quad & \left|\frac{a_n^2}{b_n^2} - 1\right| \sup_{x\in\mathbb{R}_0} \frac{x^2}{1+x^2} + \frac{2a_n}{b_n^2}\left|\lambda + 2 + \frac{g'(1)}{g(1)}\right| \sup_{x\in\mathbb{R}_0} \frac{x}{1+x^2} \\ & + \frac{1}{b_n^2}\left|(\lambda+1)(\lambda+2) + (2\lambda+4)\frac{g'(1)}{g(1)} + \frac{g''(1)}{g(1)}\right| \sup_{x\in\mathbb{R}_0} \frac{1}{1+x^2}. \end{aligned}$$

So we have

$$\lim_{n\to\infty} \left\|L_n^*(t^2;x) - x^2\right\|_\rho = 0.$$

Therefore, the desired result follows from Theorem 2.6. □

Theorem 2.8. *If $f \in C_\rho^*$, then the inequality*

$$\sup_{x\geq 0} \frac{\left|L_n^*(f;x) - f(x)\right|}{(1+x^2)^3} \leq K\Omega(f; \frac{1}{\sqrt{b_n}})$$

is satisfied for a sufficiently large n, where K is a constant independent of a_n, b_n.

Proof. From (1.12), we write

$$\begin{aligned}\left|L_n^*(f;x)-f(x)\right| &\leq 2(1+\delta_n^2)\Omega(f;\delta_n)(1+x^2)\frac{e^{-a_nx}}{g(1)}\\ &\quad\times\sum_{k=0}^{\infty}p_k(a_nx)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-b_nt}t^{\lambda+k}(1+\frac{|t-x|}{\delta_n})(1+(t-x)^2)dt\\ &\leq 4\Omega(f;\delta_n)(1+x^2)\left\{1+\frac{1}{\delta_n}\frac{e^{-a_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(a_nx)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-b_nt}t^{\lambda+k}\left|t-x\right|dt\right.\\ &\quad+\frac{e^{-a_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(a_nx)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-b_nt}t^{\lambda+k}(t-x)^2dt\\ &\quad\left.+\frac{1}{\delta_n}\frac{e^{-a_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(a_nx)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-b_nt}t^{\lambda+k}\left|t-x\right|(t-x)^2dt\right\}\end{aligned}$$

for any $\delta_n > 0$. Applying Cauchy-Schwarz inequality , we get

$$\left|L_n^*(f;x)-f(x)\right| \leq 4\Omega(f;\delta_n)(1+x^2)\left(1+\frac{2}{\delta_n}\sqrt{\phi_1}+\phi_1+\frac{1}{\delta_n}\sqrt{\phi_1\phi_2}\right) \tag{2.10}$$

where $\phi_1 = L_n^*((t-x)^2;x)$ and $\phi_2 = L_n^*((t-x)^4;x)$. By simple calculation, we have

$$\begin{aligned}L_n^*(t^3;x) &= \frac{a_n^3}{b_n^3}x^3+\frac{3a_n^2}{b_n^3}\left(\lambda+3+\frac{g'(1)}{g(1)}\right)x^2\\ &+\frac{3a_n}{b_n^3}\left((\lambda+2)(\lambda+3)+2(\lambda+3)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)x\\ &+\frac{1}{b_n^3}\left((\lambda+1)(\lambda+2)(\lambda+3)+3(\lambda+2)(\lambda+3)\frac{g'(1)}{g(1)}+3(\lambda+3)\frac{g''(1)}{g(1)}+\frac{g'''(1)}{g(1)}\right),\end{aligned} \tag{2.11}$$

$$\begin{aligned}L_n^*(t^4;x) &= \frac{a_n^4}{b_n^4}x^4+\frac{4a_n^3}{b_n^4}\left(\lambda+4+\frac{g'(1)}{g(1)}\right)x^3\\ &+\frac{6a_n^2}{b_n^4}\left((\lambda+3)(\lambda+4)+2(\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)x^2\\ &+\frac{4a_n}{b_n^4}\left((\lambda+2)(\lambda+3)(\lambda+4)+3(\lambda+3)(\lambda+4)\frac{g'(1)}{g(1)}+3(\lambda+4)\frac{g''(1)}{g(1)}+\frac{g'''(1)}{g(1)}\right)x\\ &+\frac{1}{b_n^4}\left((\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)+4(\lambda+2)(\lambda+3)(\lambda+4)\frac{g'(1)}{g(1)}\right.\\ &\left.+6(\lambda+3)(\lambda+4)\frac{g''(1)}{g(1)}+4(\lambda+4)\frac{g'''(1)}{g(1)}+\frac{g^{(iv)}(1)}{g(1)}\right)\end{aligned} \tag{2.12}$$

and hence from (2.1)- (2.3) and (2.11), (2.12), we have the following central moments of the operators L_n^* :

$$\begin{aligned}
\phi_1 &= L_n^*((t-x)^2;x) = \left(1-\frac{a_n}{b_n}\right)^2 x^2 + \left(\frac{2a_n}{b_n^2}\left(\lambda+2+\frac{g'(1)}{g(1)}\right) - \frac{2}{b_n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right)\right)x \\
&\quad +\frac{1}{b_n^2}\left((\lambda+1)(\lambda+2)+(2\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right) \\
\phi_2 &= L_n^*((t-x)^4;x) = \left(1-\frac{a_n}{b_n}\right)^4 x^4 \\
&\quad +4\left[\frac{a_n^3}{b_n^4}(\lambda+4+\frac{g'(1)}{g(1)}) - \frac{3a_n^2}{b_n^3}(\lambda+3+\frac{g'(1)}{g(1)}) + \frac{3a_n}{b_n^2}(\lambda+2+\frac{g'(1)}{g(1)}) - \frac{1}{b_n}(\lambda+1+\frac{g'(1)}{g(1)})\right]x^3 \\
&\quad +6\left[\frac{a_n^2}{b_n^4}((\lambda+3)(\lambda+4)+2(\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}) - \frac{2}{b_n^3}((\lambda+2)(\lambda+3)\right. \\
&\quad \left. +2(\lambda+3)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}) + \frac{1}{b_n^2}\left((\lambda+1)(\lambda+2)+(2\lambda+4)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)\right]x^2 \\
&\quad +4\left[\frac{a_n}{b_n^4}((\lambda+2)(\lambda+3)(\lambda+4)+3(\lambda+3)(\lambda+4)\frac{g'(1)}{g(1)}+3(\lambda+4)\frac{g''(1)}{g(1)}+\frac{g'''(1)}{g(1)})\right. \\
&\quad \left. -\frac{1}{b_n^3}((\lambda+1)(\lambda+2)(\lambda+3)+3(\lambda+2)(\lambda+3)\frac{g'(1)}{g(1)}+3(\lambda+3)\frac{g''(1)}{g(1)}+\frac{g'''(1)}{g(1)})\right]x \\
&\quad +\frac{1}{b_n^4}\left((\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)+4(\lambda+2)(\lambda+3)(\lambda+4)\frac{g'(1)}{g(1)}\right. \\
&\quad \left. +6(\lambda+3)(\lambda+4)\frac{g''(1)}{g(1)}+4(\lambda+4)\frac{g'''(1)}{g(1)}+\frac{g^{(iv)}(1)}{g(1)}\right).
\end{aligned}$$

Using condition (1.7), one has

$$\begin{aligned}
\phi_1 &= O(\frac{1}{b_n})(x^2+x), \\
\phi_2 &= O(\frac{1}{b_n})(x^4+x^3+x^2+x).
\end{aligned}$$

Substituting these equalities in (2.10) we get

$$\begin{aligned}
\left|L_n^*(f;x)-f(x)\right| &\leq 4\Omega(f;\delta_n)(1+x^2) \\
&\quad \times\left(1+\frac{2}{\delta_n}\sqrt{O(\frac{1}{b_n})(x^2+x)}+O(\frac{1}{b_n})(x^2+x)+\frac{1}{\delta_n}O(\frac{1}{b_n})\sqrt{(x^4+x^3+x^2+x)(x^2+x)}\right)
\end{aligned}$$

and choosing $\delta_n = \frac{1}{\sqrt{b_n}}$, for sufficiently large n, we obtain desired result. □

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